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THE ROLLING MOTION OF A BALL ON A SURFACE. NEW INTEGRALS AND HIERARCHY OF DYNAMICS

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The paper is concerned with the problem on rolling of a homogeneous ball on an arbitrary surface. New cases when the problem is solved by quadratures are presented. The paper also indicates a special case when an additional integral and invariant measure exist. Using this case, we obtain a nonholonomic generalization of the Jacobi problem for the inertial motion of a point on an ellipsoid. For a ball rolling, it is also shown that on an arbitrary cylinder in the gravity field the ball's motion is bounded and, on the average, it does not move downwards. All the results of the paper considerably expand the results obtained by E. Routh in XIX century.

Contents

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1. Introduction

In this paper we consider the problem of sliding-free rolling of a dynamically symmetrical ball (the central tensor of inertia is spherical $\mathbf{I} = \mu\mathbf{E}$) on an arbitrary surface. As it was indicated by E. Routh in his famous treatise [10], if the surface is a surface of revolution the problem is integrable even in the presence of axisymmetric potential fields. Here we give a more complete analysis of the Routh solution for the body of revolution, and present new integrals for the case of ball rolling on non-symmetrical surfaces of the second order.

2. The equations for a ball moving on a surface

In rigid body dynamics it is customary to introduce a body-fixed frame of reference. However, while studying motion of a homogeneous ball, it is more convenient to write the equations of motion with respect to a certain fixed in space frame of reference. In such a frame a balance of linear momentum and of angular momentum with respect to the ball's center of mass, involving the reaction and the external forces, may be written as

$$m\dot{\boldsymbol{v}} = \boldsymbol{N} + \boldsymbol{F}, \qquad (\mathbf{I}\boldsymbol{\omega})^{\cdot} = \boldsymbol{a} \times \boldsymbol{N} + \boldsymbol{M}_{\boldsymbol{F}}. \tag{2.1}$$

The sliding-free condition reads (the contact point velocity vanishes)

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{a} = 0. \tag{2.2}$$

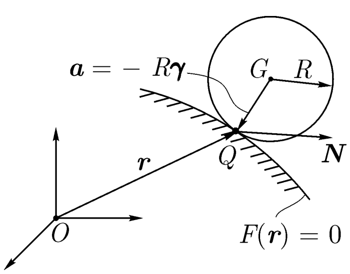

Fig. 1. The rolling of a ball on a surface (G is the center of mass, Q is a point of contact with the surface)

Here m is the mass of the ball, $\boldsymbol{v}$ isthe velocity of its center of mass, $\boldsymbol{\omega}$ is the angular velocity, $\mathbf{I} = \mu\mathbf{E}$ is the (spherical) central tensor of inertia, $\boldsymbol{a}$ is the vector from the center of mass to the point of contact, R is the ball's radius, $\boldsymbol{N}$ is the reaction at the contact point (see Fig. 1), $\boldsymbol{F}$ and $\boldsymbol{M}_F$ are the external force and the moment of forces with respect to the point of contact respectively.

Eliminating from these equations the reaction $\boldsymbol{N}$ and making use of the fact that the contact point velocities on the surface and on the ball coincide, we obtain the system of six equations:

$$\dot{\boldsymbol{M}} = D\dot{\boldsymbol{\gamma}} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) + \boldsymbol{M}_F, \quad \dot{\boldsymbol{r}} + R\dot{\boldsymbol{\gamma}} = \boldsymbol{\omega} \times R\boldsymbol{\gamma}. \tag{2.3}$$

Here $D = mR^2$. These equations govern the behaviour of the vector of the angular momentum (with respect to the contact point $\boldsymbol{M}$), and of the vector $\boldsymbol{\gamma} = -R^{-1}\boldsymbol{a}$ normal to the surface (Fig. 1). The vectors $\boldsymbol{\omega}$ and $\boldsymbol{r}$ (the position vector of the contact point) are obtained from the relations

$$\boldsymbol{M} = \mu\boldsymbol{\omega} + D\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}), \quad \boldsymbol{\gamma} = \frac{\nabla F(\boldsymbol{r})}{|\nabla F(\boldsymbol{r})|}, \tag{2.4}$$

$F(\boldsymbol{r}) = 0$ is the equation of the surface on which the ball rolls. The last equation in (2.4) defines the Gaussian map. Further on, following Routh, we will use the explicit form of the surface on which *the ball's center of mass* is moving. This surface, defined by the position vector $\boldsymbol{r}' = \boldsymbol{r} + R\boldsymbol{\gamma}$, is equidistant relative to the surface on which the contact point is moving.

In the case of potential forces, the moment $\boldsymbol{M}_F$ is expressed in terms of the potential $U(\boldsymbol{r}') = U(\boldsymbol{r} + R\boldsymbol{\gamma})$. This potential depends on the position of the ball's center of mass as follows $\boldsymbol{M}_F = R\boldsymbol{\gamma} \times \dfrac{\partial U}{\partial \boldsymbol{r}'}$.

REMARK 1. In his treatise [10] Routh obtained the equations of motion of the ball in semifixed axes and presented the cases when these equations can be analytically solved. Actually, in the majority of the subsequent publications [1, 9] Routh's results were just restated without any essential expansion. It should be noted that Routh was especially interested in the stability of particular solutions (e. g. a ball rotating around the vertical axis at the top of a surface of revolution). Here we are not giving the original form of Routh's equations. Equations (2.3) are, in many respects, similar to the equations describing an arbitrary body motion on a plane and a sphere [5]. This allows us to consider many problems (e. g. those of integrability) from the single point of view.

The integrals of motion. In the case of potential field with potential $U(\boldsymbol{r} + R\boldsymbol{\gamma})$, equations (2.3) possess the integral of energy and the geometric integral

$$H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) + U(\boldsymbol{r} + R\boldsymbol{\gamma}), \qquad F_1 = \boldsymbol{\gamma}^2 = 1. \tag{2.5}$$

In the case of an arbitrary surface $F(\boldsymbol{r}) = 0$, apart from these integrals, the system (2.3) has neither measure, nor two additional integrals, which are necessary for the integrability according to the last multiplier theory (the Euler–Jacobi theory). The system's behavior is chaotic. As it is shown later, in some cases there exists a measure and only one additional integral. In such a case chaos becomes “weaker”. As it was noted by Routh, for a surface of revolution there exist two additional integrals, the system is integrable, and its behavior is regular. The reduced system becomes a Hamiltonian one after an appropriate change of time.

Rolling on a surface of the second order. We are going now to revisit equations(2.3) for the case when the ball's center of mass is moving on a surface of the second order. The surface is defined by the equation

$$\big(\boldsymbol{r} + R\boldsymbol{\gamma}, \mathbf{B}^{-1}(\boldsymbol{r} + R\boldsymbol{\gamma})\big) = 1, \qquad \mathbf{B} = \operatorname{diag}(b_1, b_2, b_3) \tag{2.6}$$

(for an ellipsoid, $b_i > 0$ and are equal to the squares of the main semi-axes). Solving (2.6) for the position vector $\boldsymbol{r}$ gives

$$\boldsymbol{r} + R\boldsymbol{\gamma} = \frac{\mathbf{B}\boldsymbol{\gamma}}{\sqrt{(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})}}. \tag{2.7}$$

We obtain the equations of motion in terms of the variables $\boldsymbol{M}, \boldsymbol{\gamma}$:

$$\dot{\boldsymbol{M}} = -\frac{D}{\mu + D}(\boldsymbol{M}, \dot{\boldsymbol{\gamma}})\boldsymbol{\gamma}, \qquad \dot{\boldsymbol{\gamma}} = \frac{R\sqrt{(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})}}{\mu + D}\boldsymbol{\gamma} \times \big(\boldsymbol{\gamma} \times \mathbf{B}^{-1}(\boldsymbol{\gamma} \times \boldsymbol{M})\big) \tag{2.8}$$

A ball on a rotating surface. Let us also consider the motion a ball on a surface rotating with constant angular velocity $\boldsymbol{\Omega}$. One particular case of this problem (a plane and a sphere) was also investigated by Routh [10]. By analogy with the previous case, replacing the nonholonomic relation (2.2) by

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{a} = \boldsymbol{\Omega} \times \boldsymbol{r}, \tag{2.9}$$

we get

$$\begin{gathered} \dot{\boldsymbol{M}} = m\big(\dot{\boldsymbol{a}} \times (\boldsymbol{\omega} \times \boldsymbol{a}) + \boldsymbol{a} \times (\boldsymbol{\Omega} \times \dot{\boldsymbol{r}})\big) + \boldsymbol{M}_F \\ \dot{\boldsymbol{r}} + R\dot{\boldsymbol{\gamma}} = \boldsymbol{\omega} \times R\dot{\boldsymbol{\gamma}} + \boldsymbol{\Omega} \times \boldsymbol{r}. \end{gathered} \tag{2.10}$$

Here $\boldsymbol{a} = -R\boldsymbol{\gamma}$.

It should be noted that in the case of an arbitrary surface the system of six equations (2.10) is not closed because the position vector of the point on the surface, $\boldsymbol{r}$, is not expressed in terms of $\boldsymbol{\gamma}$ only; one should also introduce the equation for the angle of rotation of the surface around the fixed axis. Nevertheless, if the surface is axisymmetric, and the symmetry axis coincides with the axis of rotation, equations (2.10) get closed. We assume this to be fulfilled in further text.

Equations (2.2), (2.10) are in many respects similar to the equations, defining the rigid body rolling on a plane or a sphere, which we thoroughly investigated in the paper [5]. We shall use this fact while transferring the corresponding results onto systems (2.2), (2.10).

3. Ball's motion on a surface of revolution

First of all, let us consider the cases of integrability of equations (2.3), (2.10), associated with the rotational symmetry of the surface on which the ball rolls. Here we are using the technique introduced in [5] and concerned with the analysis of a certain reduced system in terms of new variables K_1, K_2, K_3, γ_3. We shall also assume the surface to be rotating around the axis of symmetry with constant angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$, $\Omega \neq 0$. The particular case, $\Omega = 0$, was discussed by Routh, who obtained the majority of the following results (although he missed some cases of equal interest).

The equation of a surface of revolution with respect to an immovable reference frame may be written as

$$\begin{gathered} r_1 = (f(\gamma_3) - R)\gamma_1, \quad r_2 = (f(\gamma_3) - R)\gamma_2, \\ r_3 = \int \Big(f(\gamma_3) - \frac{1-\gamma_3^2}{\gamma_3} f'(\gamma_3) \Big) d\gamma_3 - R\gamma_3, \end{gathered} \tag{3.1}$$

where $f(\gamma_3)$ is a certain function specifying the surface parametrization. The parametrization (3.1) is to be so chosen as to make the form of the reduced system as simple as possible.

In the case being considered equations (2.3), (2.10) allow an invariant measure with density

$$\rho = \big(f(\gamma_3)\big)^3 g(\gamma_3), \quad \text{where} \quad g(\gamma_3) = f(\gamma_3) - \frac{1-\gamma_3^2}{\gamma_3} f'(\gamma_3). \tag{3.2}$$

Apart from the invariant measure, the equations also have a simple symmetry field

$$\hat{\boldsymbol{v}} = M_1 \frac{\partial}{\partial M_2} - M_2 \frac{\partial}{\partial M_1} + \gamma_1 \frac{\partial}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \gamma_1}, \tag{3.3}$$

which is caused by the rotational symmetry.

In our book [3] we make a frequent use of the fact that to obtain the simplest form of the reduced system in the presence of symmetries, one needs to choose the most relevant *integrals of the field of symmetries* which, in this case, may be written as

$$K_1 = (\boldsymbol{M}, \boldsymbol{\gamma}) f(\gamma_3), \quad K_2 = \mu\omega_3 = \frac{\mu M_3 + D(\boldsymbol{M}, \boldsymbol{\gamma})\gamma_3}{\mu + D}, \quad K_3 = \frac{M_1\gamma_2 - M_2\gamma_1}{\sqrt{1-\gamma_3}}, \quad \gamma_3. \tag{3.4}$$

In terms of the chosen variables, the reduced system takes the form

$$\begin{gathered} \dot{\gamma}_3 = kK_3, \\ \dot{K}_1 = kK_3 \Big(\frac{f'}{\gamma_3} K_2 + \Big(1 - \frac{f}{R}\Big) g\mu\Omega \Big), \\ \dot{K}_2 = kK_3 \frac{D}{\mu + D} \Big(\frac{1}{f} K_1 - \gamma_3 \Big(1 - \frac{g}{R}\Big) \mu\Omega \Big), \\ \dot{K}_3 = -k \frac{(\mu + D) g}{\mu^2 (1-\gamma_3^2) f^2} \Big(\frac{\gamma_3 K_1 - f K_2}{f(1-\gamma_3^2)} ((\mu + D\gamma_3^2)K_1 - \gamma_3 f(\mu + D)K_2) + \\ + \Big(1 - \frac{g}{R}\Big) ((\mu + 2D\gamma_3^2)K_1 - \gamma_3 f(\mu + 2D)K_2) \Big) \mu\Omega, \end{gathered} \tag{3.5}$$

where $k = \dfrac{R\sqrt{1-\gamma_3^2}}{(\mu+D)g(\gamma_3)}$. It is easy to show that this system of equations possesses an invariant measure with density $\rho = k^{-1}$. The system (3.5) can be explicitly integrated in the following way.

Let us divide the second and the third equation of system (3.5) by $\dot{\gamma}_3$. Then we obtain the nonautonomous system of two linear equations with the independent variable γ_3

$$\frac{dK_1}{d\gamma_3} = \frac{f'}{\gamma_3}K_2 + \Big(1-\frac{f}{R}\Big)g\mu\Omega, \quad \frac{dK_2}{d\gamma_3} = \frac{D}{\mu+D}\Big(\frac{1}{f}K_1 - \gamma_3\Big(1-\frac{g}{R}\Big)\mu\Omega\Big). \tag{3.6}$$

This system of linear equations always possesses two integrals which are linear in K_1, K_2. The coefficients in the integrals are functions of γ_3, and in the general case cannot be obtained in the explicit (algebraic) form. Having divided the last equation of the system (3.5) by γ_3 and substituting into this equation the known solution of the system (3.6), we obtain the explicit quadrature for $K_3(\gamma_3)$. Using the first equation from (3.5) we can obtain the expression for $\gamma_3(t)$.

In the case $\Omega = 0$, the system (3.5) has the integral of energy

$$H = (\boldsymbol{M},\boldsymbol{\omega}) = \frac{1}{2}\left(\frac{K_1^2}{\Lambda f^2} + \frac{(\mu+D)(\gamma_3K_1 - fK_2)^2}{\mu^2 f^2(1-\gamma_3^2)} + \frac{K_3^2}{\mu+D}\right). \tag{3.7}$$

The quadrature for $\gamma_3(t)$ may be obtained upon substitution of $K_3 = k^{-1}\dot{\gamma}_3$ into (3.7).

For the reduced system (3.5) the following theorem is valid:

Theorem 1. *For $\Omega = 0$ by the change of time $k\,dt = d\tau$, the system* (3.5) *can be represented in the Hamiltonian form*

$$\frac{dx_i}{d\tau} = \{x_i, H\}, \quad \boldsymbol{x} = (\gamma_3, K_1, K_2, K_3). \tag{3.8}$$

The bracket is degenerate and specified by the relations

$$\{\gamma_3, K_3\} = \mu + D, \quad \{K_1, K_3\} = (\mu+D)\frac{f'}{\gamma_3}K_2, \quad \{K_2, K_3\} = \frac{D}{f}K_1 \tag{3.9}$$

(*for the other pairs of variables the brackets are zero*).

Proof is a straightforward exercise consisting in derivation of (3.5) ($\Omega = 0$) from (3.8) and verification of the Jacobi identity for the bracket (3.9). ■

One can assert that the equations for the linear integrals of the system (3.6) exactly coincide with the equations for the Casimir function of the bracket (3.9).

It should also be noted that for $\Omega = 0$ the system (3.5) has "skew-symmetrical notation", similar to the Hamiltonian form (3.8), (3.9)

$$\frac{dx_i}{d\tau} = J_{\lambda ij}(x)\frac{\partial H}{\partial x_j}; \quad J_{ij} = -J_{ji}. \tag{3.10}$$

The matrix $\mathbf{J}_\lambda$ is of a somewhat more general form than that of the corresponding bracket (3.9)

$$\mathbf{J}_\lambda = \begin{pmatrix} 0 & 0 & 0 & \mu+D \\ 0 & 0 & \lambda & (\mu+D)\dfrac{f'}{\gamma_3}K_2 + \lambda u \\ 0 & -\lambda & 0 & \dfrac{D}{f}K_1 + \lambda v \\ -(\mu+D) & -(\mu+D)\dfrac{f'}{\gamma_3}K_2 - \lambda u & \dfrac{D}{f}K_1 + \lambda v & 0 \end{pmatrix},$$
$$u = \frac{(\mu+D)^2(\gamma_3K_1 - fK_2)}{\mu^2 f(1-\gamma_3^2)K_3}, \quad v = \frac{(\mu+D)\big((\mu+D\gamma_3^2)K_1 - f\gamma_3(\mu+D)K_2\big)}{\mu^2 f^2(1-\gamma_3^2)K_3}. \tag{3.11}$$

Here λ is an arbitrary function of γ_3, K_i. For $\lambda = 0$, one again gets bracket (3.9), and the tensor $\mathbf{J}_\lambda$ satisfies the Jacobi identity. However, in the general case, $\mathbf{J}_\lambda$ does not satisfy the Jacobi identity (it is not of the Poisson type). Nevertheless, if we assume

$$\lambda = w(\gamma_3)K_3, \tag{3.12}$$

then the tensor $\lambda^{-1}\mathbf{J}_\lambda$ is a Poisson one, and the quantity λ is a reducing multiplier (according to Chaplygin).

Thus, we obtain the following Hamiltonian vector field

$$\boldsymbol{v}_\lambda = (\lambda^{-1}\mathbf{J}_\lambda)\nabla H,$$

for which $\operatorname{div} \boldsymbol{v}_\lambda \neq 0$ ($\operatorname{div} \lambda\boldsymbol{v}_\lambda = 0$). The examples of Hamiltonian fields with nonzero divergency were almost not discussed earlier. One particular case of the Poisson tensor $\lambda^{-1}\mathbf{J}_\lambda$ was found by J. Hermans in [12]. Hermans used his own system of reduced variables which slightly differs from ours.

To clarify the behavior of a ball on a surface of revolution, we will discuss the cases, when this surface is a paraboloid, a sphere, a cone, and a cylinder. These problems were investigated by Routh in [10] for $\Omega = 0$. Here we will expand the results for $\Omega \neq 0$.

A paraboloid of revolution. Suppose the ball's center of mass is moving on the paraboloid of revolution $z = c(x^2 + y^2)$. In equations (3.1) we, therefore, put

$$f(\gamma_3) = -\frac{1}{2c\gamma_3}. \tag{3.13}$$

The density of the invariant measure (3.2) (up to an unessential factor) can be written as

$$\rho = \frac{1}{\gamma_3^6}. \tag{3.14}$$

In this case, the two-dimensional system (3.6) reads

$$\frac{dK_1}{d\gamma_3} = \frac{1}{2c\gamma_3^3}\Big(K_2 - \frac{1 + 2cR\gamma_3}{2cR\gamma_3}\mu\Omega\Big), \quad \frac{dK_2}{d\gamma_3} = -\frac{D}{\mu + D}\Big(2c\gamma_3 K_1 + \frac{(1 + 2cR\gamma_3^3)}{2cR\gamma_3^2}\mu\Omega\Big). \tag{3.15}$$

For the variable K_2 we obtain a *homogeneous* second-order linear equation whose coefficients are homogeneous functions of γ_3 (at $\Omega \neq 0$)

$$K_2'' - \frac{1}{\gamma_3}K_2' + \frac{D}{\mu + D}\frac{1}{\gamma_3^2}K_2 = \frac{D(2 + cR\gamma_3)}{(\mu + D)cR\gamma_3^3}\mu\Omega.$$

Its general solution may be represented as γ_3^α, $\alpha = \mathrm{const}$

$$K_2 = c_1\gamma_3^{1-\nu} - c_2\gamma_3^{1+\nu} + \mu\Omega\Big(1 + \frac{2D}{cR(\mu + 4D)}\frac{1}{\gamma_3}\Big), \quad \nu^2 = \frac{\mu}{\mu + D}. \tag{3.16}$$

For the variable K_1 from (3.15) we, similarly, obtain

$$K_1 = \frac{\mu + D}{2cD}\Big(-(1 - \nu)\gamma_3^{-1-\nu}c_1 + (1 + \nu)\gamma_3^{-1+\nu}c_2\Big) - \frac{\mu\Omega}{2c}\Big(1 - \frac{\mu}{2cR(3\mu + 4D)}\frac{1}{\gamma_3^3}\Big). \tag{3.17}$$

Solving for the constants c_1, c_2 from (3.16), (3.17), we obtain the integrals of the system (3.6). These integrals are linear in to K_1, K_2 and have form

$$\begin{aligned} F_2 &= \frac{D}{2\sqrt{\mu(\mu + D)}}\gamma_3^{\nu}\Big(2c\gamma_3K_1 + \frac{\mu + D}{D\gamma_3}K_2 + \mu\Omega\Big(\gamma_3 - \frac{(\mu + D)(1 + \nu)}{D\gamma_3} - \frac{2 + \nu}{2cR(2 - \nu)\gamma_3^2}\Big)\Big) \\ F_3 &= \frac{D}{2\sqrt{\mu(\mu + D)}}\gamma_3^{-\nu}\Big(2c\gamma_3K_1 + \frac{\mu + D}{D\gamma_3}K_2 + \mu\Omega\Big(\gamma_3 - \frac{(\mu + D)(1 - \nu)}{D\gamma_3} - \frac{2 - \nu}{2cR(2 + \nu)\gamma_3^2}\Big)\Big) \end{aligned} \tag{3.18}$$

The product F_2F_3 gives *an algebraic quadratic integral*.

For $\Omega=0$ the equations also have the integral of energy

$$H=\frac{2c^2\gamma_3^2}{\mu}K_1^2+\frac{1}{2}\frac{\mu+D}{\mu^2(1-\gamma_3^2)}\big(2c\gamma_3^2K_1+K_2\big)^2+\frac{1}{2}\frac{K_3^2}{\mu+D}. \tag{3.19}$$

Remark 2. Some forms of surfaces (on which a ball is rolling) are investigated in the paper [6] (details are given below). In this paper it is shown that if a ball is rolling on a paraboloid of revolution, then the system (2.3) is reduced to a particular class of Fuchsian equations. Routh himself considered the case when not the contact point, but the center of mass is moving on a paraboloid. We should note again that in this case equations (2.3) have algebraic integrals. The motion of a homogeneous ball on a surface of revolution was also studied by F. Noether [11].

An axisymmetric ellipsoid. Consider the motion of a dynamically symmetrical ball when its center of mass is moving on a fixed ellipsoid ($\Omega=0$). In this case

$$f(\gamma_3)=\frac{b_1}{\sqrt{b_1(1-\gamma_3^2)+b_3\gamma_3^2}}, \tag{3.20}$$

where b_1, b_2 are the squares of the ellipsoid principal semi-axes. The density of the invariant measure (3.2) (up to a constant factor) is

$$\rho=\big(b_1(1-\gamma_3^2)+b_3\gamma_3^2\big)^{-3}. \tag{3.21}$$

In this case the variables N_1, N_2, K_3 are more convenient than the variables (3.4). Here

$$N_1=(\boldsymbol{M},\boldsymbol{\gamma}),\quad N_2=\frac{\mu}{\mu+D}f(\gamma_3)\big(\gamma_3(\boldsymbol{M},\boldsymbol{\gamma})-M_3\big), \tag{3.22}$$

which satisfy the system, similar to (3.5) for $\Omega=0$

$$\begin{gathered}\dot N_1=-kK_3\frac{f'}{\gamma_3f^2}N_2,\quad \dot N_2=kK_3\frac{\mu}{\mu+D}fN_1,\\ \dot K_3=-k\frac{(\mu+D)g}{\mu^2\gamma_3(1-\gamma_3^2)^2f^3}N_2\big(\mu f(1-\gamma_3^2)N_1+(\mu+D)\gamma_3N_2\big),\end{gathered} \tag{3.23}$$

where $k=\dfrac{R\sqrt{1-\gamma_3^2}}{(\mu+D)g}$.

Using (3.20), we obtain two linear equations with independent variable γ_3

$$\frac{dN_1}{d\gamma_3}=-\frac{(b_1-b_3)}{b_1\sqrt{b_1(1-\gamma_3^2)+b_3\gamma_3^2}}N_2,\quad \frac{dN_2}{d\gamma_3}=\frac{\mu b_1}{(\mu+D)\sqrt{b_1(1-\gamma_3^2)+b_3\gamma_3^2}}N_1. \tag{3.24}$$

It is easy to show that system (3.24) has a quadratic integral with constant coefficients

$$F_2=b_1^2\frac{\mu}{\mu+D}N_1^2+(b_1-b_3)N_2^2. \tag{3.25}$$

Below we will show that this integral may be generalized to the case of a three-axial ellipsoid.

The system (3.24) can be solved in terms of elementary functions. Depending on the sign of the difference b_1-b_3, the solution may be written as:

1. $\boldsymbol{b_1 > b_3}$, $\boldsymbol{a^2 = \dfrac{b_1}{b_1 - b_3} > 1}$.

$$N_1 = c_1 \sin\varphi(\gamma_3) + c_2\cos\varphi(\gamma_3), \quad N_2 = a\sqrt{\frac{\mu b_1}{\mu + D}}\big(-c_1\cos\varphi(\gamma_3) + c_2\sin\varphi(\gamma_3)\big),$$
$$\varphi(\gamma_3) = \nu\,\mathrm{arctg}\,\frac{\gamma_3}{\sqrt{a^2-\gamma_3^2}}, \qquad \nu = \sqrt{\frac{\mu}{\mu+D}}. \tag{3.26}$$

2. $\boldsymbol{b_1 < b_3}$, $\boldsymbol{a^2 = \dfrac{b_1}{b_3 - b_1} > 0}$.

$$N_1 = c_1\tau^{-\nu} + c_2\tau^{\nu}, \quad N_2 = a\sqrt{\frac{\mu b_1}{\mu+D}}\Big(-c_1\tau^{-\nu} + c_2\tau^{\nu}\Big),$$
$$\tau(\gamma_3) = \gamma_3 + \sqrt{a^2+\gamma_3^2}, \qquad \nu = \sqrt{\frac{\mu}{\mu+D}}. \tag{3.27}$$

Here $c_1, c_2 = \mathrm{const}$. Solving for these constants, we can obtain linear integrals of motion.

Historical comments. It is an interesting fact that neither Routh, nor his followers succeeded in obtaining the simplest reduced equations (like (3.24)) and solving the problem of a ball rolling on an ellipsoid of revolution in terms of elementary functions. To integrate the equations one must appropriately choose the reduced variables such as (3.22).

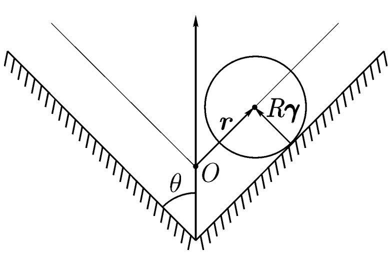

Fig. 2

A circular cone. In this case, due to the fact that the Gaussian map $\boldsymbol{\gamma} = \dfrac{\nabla F}{|\nabla F|}$ is degenerate, one should use the components of the vector $\boldsymbol{r}$ (the position vector of the ball center) as the positional variables in equations (2.10). For the cone θ (Fig. 2) we have

$$\gamma_3 = \cos\theta = \mathrm{const}, \quad \gamma_1 = \frac{k^2}{\sqrt{1+k^2}}\frac{r_1}{r_3}, \quad \gamma_2 = \frac{k^2}{\sqrt{1+k^2}}\frac{r_2}{r_3},$$
$$r_3 = k\sqrt{r_1^2+r_2^2}, \qquad k = \mathrm{tg}\,\theta. \tag{3.28}$$

The measure of equations (2.3), (2.10) in which $\boldsymbol{\gamma}$ is expressed in terms of $\boldsymbol{r}$ in accordance with (3.28), can be written in explicit form:

$$\rho = \frac{\left(\dfrac{k^2}{1-k^2}R_0 + r_3\right)^3}{r_3^2}, \tag{3.29}$$

For the reduced system let us choose the variables

$$\sigma_1 = \omega_3 + \frac{D\Omega}{\sqrt{1+k^2}\sqrt{\mu+D}}\frac{r_3}{R},$$
$$\sigma_2 = \left(r_3 + \frac{k^2}{\sqrt{1+k^2}}R\right)\left(\left(\boldsymbol{M} - \frac{k^2\mu^2}{\mu+D}\boldsymbol{\Omega}\right), \boldsymbol{\gamma}\right). \tag{3.30}$$

In terms of these variables we obtain the equations

$$\frac{d\sigma_1}{dr_3} = 0, \quad \frac{d\sigma_2}{dr_3} = \sqrt{1+k^2}\sigma_1 + \frac{\mu\Omega}{\mu+D}\left(\frac{r_3}{R} - k^2\right).$$

From these equations the following integrals can be easily obtained:

$$F_2 = \sigma_1, \quad F_3 = \sqrt{1+k^2} r_3 \sigma_1 - \sigma_2 + \frac{\mu\Omega}{\mu + D}\Big(\frac{r_3^2}{2R} - k^2 r_3\Big). \tag{3.31}$$

A circular cylinder. *The motion of a ball within a cylinder* is a well-known problem which is usually used to illustrate some unrealistic conclusions, derived by means of nonholonomic mechanics. It can be shown that a homogeneous ball, moving within a vertical cylinder, on the average, is not rolling downwards due to the gravity force. Nevertheless, this physical fact may be observed while playing basketball, when the ball has almost hit the basket, but then rapidly jumps out of it, suddenly lifting upwards. The addition of a viscous friction to this nonholonomic system, which leads to a vertical drift, is analyzed in [8], where the explicit solution of this problem has also been obtained.

For a cylinder, $\boldsymbol{\gamma} = \left(-\frac{r_1}{R_c}, -\frac{r_2}{R_c}, 0\right)$, where R_c is the cylinder radius (Fig. 3), in terms of the variables $(\boldsymbol{M}, \boldsymbol{r})$ or $(\boldsymbol{\omega}, \boldsymbol{r})$, the invariant measure's density is constant. The kinetic energy looks like

$$H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) = \frac{1}{2(\mu + D)}\left\{(M_1^2 + M_2^2) + \frac{D}{\mu}(\boldsymbol{M}, \boldsymbol{\gamma})^2 + M_3^2\right\} =$$
$$= \frac{1}{2}(\mu(\boldsymbol{\omega}, \boldsymbol{\gamma})^2 + (\mu + D)(\omega_3^2 + (\omega_1\gamma_2 - \omega_2\gamma_1)^2)).$$

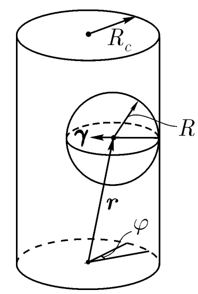

Fig. 3

The reduced system in terms of the variables $\sigma_1 = \omega_3$, $\sigma_2 = (\boldsymbol{\omega}, \boldsymbol{\gamma})$ looks like

$$\sigma_1' = \omega_3' = 0, \qquad \sigma_2' = \frac{R\omega_3 - R_c\Omega}{(R_c - R)R}. \tag{3.32}$$

Thus, we have two integrals

$$\omega_3 = \text{const}, \qquad (\boldsymbol{\omega}, \boldsymbol{\gamma}) - \frac{R\omega_3 - R_c\Omega}{R_c - R}\frac{r_3}{R} = \text{const}. \tag{3.33}$$

On writing $\widetilde{\boldsymbol{\omega}} = (\omega_1, \omega_2, \frac{R\omega_3 - R_c\Omega}{R_c - R})$, the second integral takes the form

$$(\widetilde{\boldsymbol{\omega}}, \boldsymbol{r}) = \text{const},$$

and the kinetic energy becomes

$$2H = \mu\left((\widetilde{\boldsymbol{\omega}}, \boldsymbol{\gamma}) + \widetilde{\omega}_3\frac{r_3}{R}\right)^2 + (\mu + mR^2)\widetilde{\omega}_3^2 + (\mu + mR^2)\frac{\dot{r}_3^2}{R^2}.$$

Hence, the variable r_3 (responsible for the vertical displacement of the ball) can be easily found to be

$$r_3 = -\frac{(\widetilde{\boldsymbol{\omega}}, \boldsymbol{\gamma})}{\widetilde{\omega}_3} \pm \sqrt{\frac{2H - (\mu + mR^2)\widetilde{\omega}_3^2}{\mu\widetilde{\omega}_3^2}}\sin\Big(\widetilde{\omega}_3\sqrt{\frac{\mu}{\mu + mR^2}}(t - t_0)\Big),$$

where t_0 is a constant depending on the initial conditions. It is clear that the average displacement of the ball equals zero even in the presence of the gravity field (see formula (5.6)).

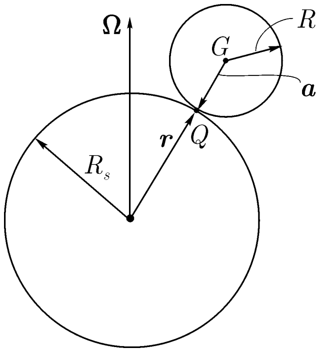

Fig. 4

Let us now consider two different variants of problem concerning the rolling of a ball on a sphere. These problems were integrated and analyzed by Routh [10].

A ball on a rotating sphere. Consider a sphere of radius R_s rotating around a certain axis with a constant angular velocity $\boldsymbol{\Omega}$. Let R be the ball radius, $\boldsymbol{a} = -R\boldsymbol{\gamma}$, $\boldsymbol{r} = R_s\boldsymbol{\gamma}$ (Fig. 4). The equations of motion in the potential field with potential $V(\boldsymbol{\gamma})$ may be represented as

$$D_1\dot{\boldsymbol{\omega}} = \frac{DR}{R_s+R}(\boldsymbol{\omega},\boldsymbol{\gamma})\boldsymbol{\omega}\times\boldsymbol{\gamma} + \frac{R_sR}{R_s+R}(\boldsymbol{\Omega},\boldsymbol{\gamma})(R\boldsymbol{\omega}+R_s\boldsymbol{\Omega})\times\boldsymbol{\gamma} + \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}},$$
$$\dot{\boldsymbol{\gamma}} = \frac{(R\boldsymbol{\omega}+R_s\boldsymbol{\Omega})\times\boldsymbol{\gamma}}{R_s+R} \qquad (3.34)$$

where $D = mR^2$, $D_1 = \mu + D$, and μ is the moment of inertia of the ball. Whether the ball rolls outside or inside the sphere is determined by the sign of R. Using the angular momentum vector $\boldsymbol{M} = D_1\boldsymbol{\omega} - D\boldsymbol{\gamma}(\boldsymbol{\omega},\boldsymbol{\gamma})$, the first equation of (3.34) may be written as

$$\dot{\boldsymbol{M}} = \frac{R_sR}{R_s+R}\Big(R((\boldsymbol{\omega}\times\boldsymbol{\gamma})(\boldsymbol{\Omega},\boldsymbol{\gamma}) + \boldsymbol{\gamma}(\boldsymbol{\Omega},\boldsymbol{\omega}\times\boldsymbol{\gamma})) - R_s(\boldsymbol{\Omega},\boldsymbol{\gamma})(\boldsymbol{\gamma}\times\boldsymbol{\Omega})\Big) + \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}}. \qquad (3.35)$$

Using $\widetilde{\boldsymbol{\omega}} = \boldsymbol{\omega} + \frac{R_s}{R}\boldsymbol{\Omega}$ in (3.34), we get

$$\begin{cases} D_1\dot{\widetilde{\boldsymbol{\omega}}} = \dfrac{DR}{R_s+R}(\widetilde{\boldsymbol{\omega}},\boldsymbol{\gamma})(\widetilde{\boldsymbol{\omega}}\times\boldsymbol{\gamma}) + \boldsymbol{\gamma}\times\dfrac{\partial V}{\partial\boldsymbol{\gamma}} \\ \dot{\boldsymbol{\gamma}} = \dfrac{R}{R_s+R}\widetilde{\boldsymbol{\omega}}\times\boldsymbol{\gamma}. \end{cases} \qquad (3.36)$$

This coincides with the original system (3.34) for $\boldsymbol{\Omega} = 0$. Thus, it will be enough to consider only the case when $\boldsymbol{\Omega} = 0$. In terms of the variables $(\boldsymbol{M},\boldsymbol{\gamma})$ equations (3.34) for $\boldsymbol{\Omega} = 0$ are written as

$$\begin{cases} \dot{\boldsymbol{M}} = \boldsymbol{\gamma}\times\dfrac{\partial V}{\partial\boldsymbol{\gamma}} \\ \dot{\boldsymbol{\gamma}} = \dfrac{R}{R_s+R}\boldsymbol{\omega}\times\boldsymbol{\gamma}. \end{cases} \qquad (3.37)$$

They possess the integrals

$$H = \frac{1}{2}(\boldsymbol{M},\boldsymbol{\omega}) + V(\boldsymbol{\gamma}), \quad F_1 = (\boldsymbol{M},\boldsymbol{\gamma}) = c, \quad F_2 = \boldsymbol{\gamma}^2 = 1. \qquad (3.38)$$

The change of time $t \to -\frac{R}{R_s+R}\tau$ transforms the second equation of (3.37) into an ordinary Poisson equation $\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}$, and the potential is multiplied by some nonessential factor. Thus we set a system (3.34) for which we know the cases when it is integrable. For example, if $V = \frac{1}{2}(\boldsymbol{\gamma},\mathbf{C}\boldsymbol{\gamma})$, $\mathbf{C} = \operatorname{diag}(c_1, c_2, c_3)$, one gets famous Neumann problem of the motion of a point on a sphere in the quadratic potential (equations (3.34) incorporate even more general situation, when $(\boldsymbol{M},\boldsymbol{\gamma}) = c \neq 0$, which corresponds to the Clebsch case [4]).

The equations (3.36), (3.37)for the motion of a ball on a sphere also manifests an analogy (discussed in [9, 10]) of problems, concerning the rolling of a homogeneous ball on a sphere in the gravity field, and the Lagrange case in the Euler–Poisson equations (the motion of a heavy dynamically symmetrical top). Indeed, for a potential of the type $V = V(\gamma_3)$, the systems (3.36), (3.37), because of the axial symmetry, have "Lagrangian integrals" $M_3 = \text{const}$ or $\omega_3 = \text{const}$.

The rolling of a ball on a rotating sphere. Now suppose that the sphere on which a ball rolls rotates freely about its center. The dynamical equations can be written as:

$$m\dot{\boldsymbol{v}} = \boldsymbol{N}, \quad \mu\dot{\boldsymbol{\omega}} = \boldsymbol{a} \times \boldsymbol{N}, \quad \mu_s\dot{\boldsymbol{\Omega}} = -\boldsymbol{r} \times \boldsymbol{N},$$
$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{a} = \boldsymbol{\Omega} \times \boldsymbol{r}.$$

Here $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$, μ, μ_s are the angular velocities and the moments of inertia of the ball and the sphere, respectively. Using the relations $\boldsymbol{r} = R_s\boldsymbol{\gamma}$, $\boldsymbol{a} = -R\boldsymbol{\gamma}$, for the quantities $\widetilde{\boldsymbol{\omega}} = \dfrac{R\boldsymbol{\omega} + R_s\boldsymbol{\Omega}}{R + R_s}$, $\boldsymbol{\gamma}$, we can write

$$(1 + D)\dot{\widetilde{\boldsymbol{\omega}}} = D\boldsymbol{\gamma} \times (\dot{\boldsymbol{\gamma}} \times \widetilde{\boldsymbol{\omega}}), \quad \dot{\boldsymbol{\gamma}} = \widetilde{\boldsymbol{\omega}} \times \boldsymbol{\gamma},$$
$$D = \frac{mR^2}{\mu} + \frac{mR_s^2}{\mu_s}. \tag{3.39}$$

By the change of time $dt \to \alpha\, dt$, $\alpha = \text{const}$, this system can be reduced to equations (3.36) and, for this reason, is integrable.

4. Rolling of a ball on surfaces of the second order

Consider the dynamics of a ball in greater detail for the case when its center of mass is moving on a surface of the second order

$$\big(\boldsymbol{r} + R\boldsymbol{\gamma}, \mathbf{B}^{-1}(\boldsymbol{r} + R\boldsymbol{\gamma})\big) = 1, \quad \mathbf{B} = \mathrm{diag}(b_1, b_2, b_3). \tag{4.1}$$

In this case the equations of motion are identical in form to (2.8).

It can be shown that these equations possess an invariant measure and a quadratic integral of the form

$$\rho = (\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})^{-2},$$
$$F_2 = \frac{(\boldsymbol{\gamma} \times \boldsymbol{M}, \mathbf{B}^{-1}(\boldsymbol{\gamma} \times \boldsymbol{M}))}{(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})}. \tag{4.2}$$

Here $\mathbf{B}$ is an arbitrary (nondegenerate) matrix.

Comment I. The invariant measure's density was found rather easily, after the authors had obtained equations (2.8) for rolling of a homogeneous ball on an ellipsoid. These equations describe the *Jacobi nonholonomic problem*. The problem's name stems from the following fact. When the ball's radius is tending to zero it seems that we get the holonomic classical problem concerning the geodesic lines on an ellipsoid (solved by Jacobi in terms of elliptic functions). Apparently, one should get convinced in the correctness of such a limiting transition, which, however, does not prevent us from using the given terminology. One can treat the integral (4.2) as a generalization of the Ioachimstal quadratic integral in the Jacobi problem. Originally, the authors found this integral by numerical experiments, using the Poincaré three-dimensional map in terms of the Andoyae–Deprit variables (L, G, H, l, g, h). These variables for nonholonomic mechanics were introduced in [5] (see also earlier paper [2]).

Fig. 5 illustrates three-dimensional sections of a phase flow in the phase space $(l, L/G, H, g)$ at a level of energy $E = \text{const}$. The secant plane is $g = \pi/2$. It is seen that the levels $F_2 = \text{const}$ make the three-dimensional chaos "foliated" into two-dimensional chaotic surfaces. The fact that the motion on the two-dimensional surfaces $F_2 = \text{const}$ ensures that no additional integrals (necessary for integrability of the problem) exist.

Let us consider various particular (may be degenerate) surfaces of the second order for which an integral of the type (4.2) exists.

An elliptic (hyperbolic) paraboloid. Let the ball's center of mass move on an elliptic paraboloid defined by the equation

$$\frac{x^2}{b_1} + \frac{y^2}{b_2} = 2z. \tag{4.3}$$

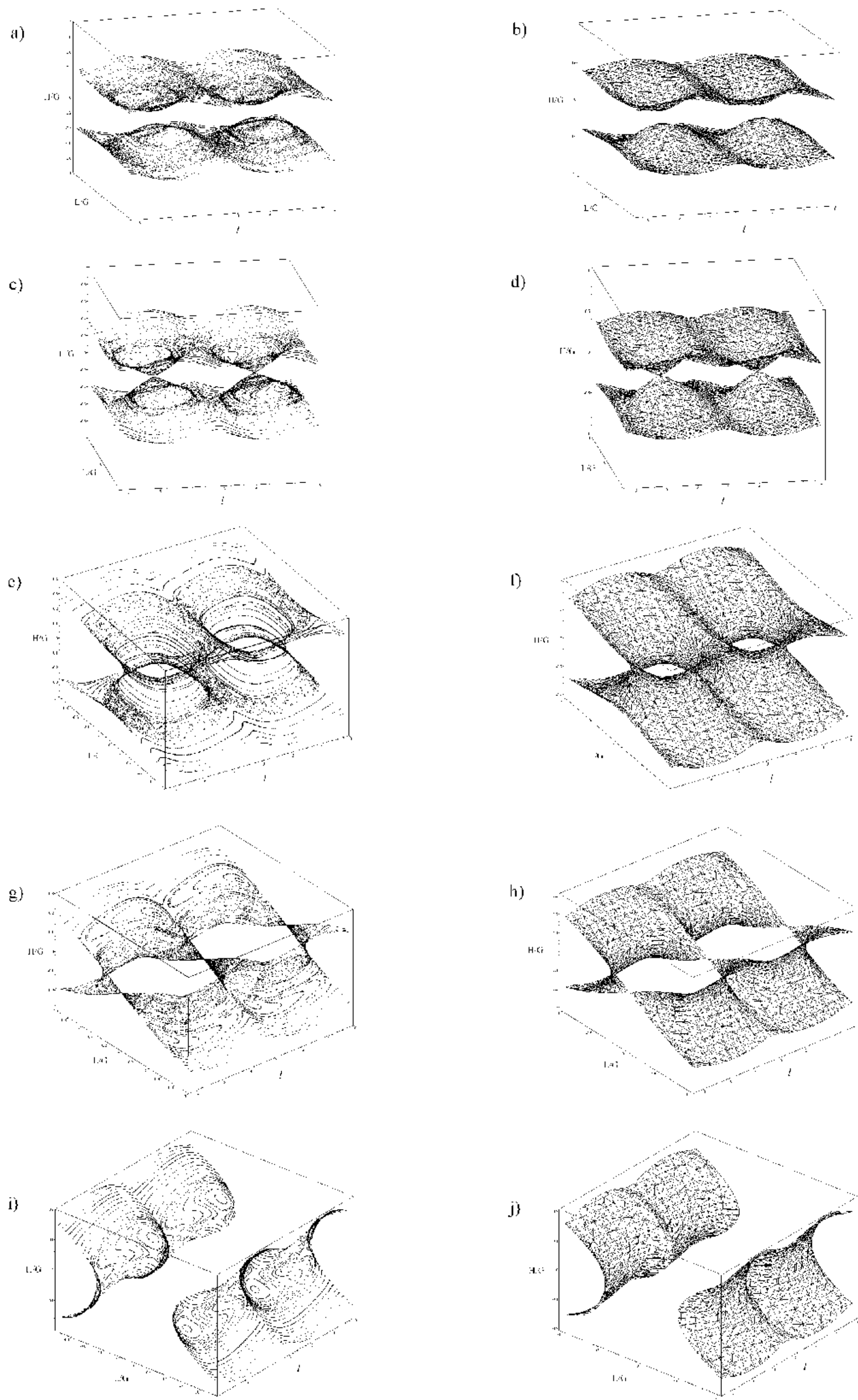

Fig. 5. Examples of three-dimensional mappings (on the left) and the corresponding level surfaces of the integral F_2 (on the right). All the mappings are constructed at a level of energy $E = 1$ for $\mathbf{B} = \mathrm{diag}(1,\, 4,\, 9)$. To the frames on the left correspond the values $F_2 = 1.7,\ 2,\ 4,\ 8,\ 10$ (from top to bottom).

Although in this case all the results may be obtained from the previous ones by means of a passage to the limit, we will derive them "of scratch".

The Gaussian map (2.7) looks like:

$$r_1 + R\gamma_1 = -b_1\frac{\gamma_1}{\gamma_3}, \quad r_2 + R\gamma_2 = -b_2\frac{\gamma_2}{\gamma_3}, \quad r_3 + R\gamma_3 = \frac{b_1\gamma_1^2 + b_2\gamma_2^2}{2\gamma_3^2}, \tag{4.4}$$

and the equations motion (2.8) take the form

$$\dot{\boldsymbol{M}} = -\frac{D}{\mu + D}(\boldsymbol{M}, \dot{\boldsymbol{\gamma}})\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \frac{R\gamma_3}{\mu + D}\boldsymbol{\gamma} \times \big(\boldsymbol{\gamma} \times \mathbf{B}^i(\boldsymbol{\gamma} \times \boldsymbol{M})\big),$$

where $\mathbf{B}^i = \operatorname{diag}(b_1^{-1}, b_2^{-1}, 0)$ is a degenerate matrix.

The invariant measure density depends on γ_3 only

$$\rho = \frac{1}{\gamma_3^4}, \tag{4.5}$$

and the quadratic integral (4.2) can be written as

$$F_2 = \frac{\big(\boldsymbol{\gamma} \times \boldsymbol{M}, \mathbf{B}^i(\boldsymbol{\gamma} \times \boldsymbol{M})\big)}{\gamma_3^2}. \tag{4.6}$$

Comment II. The integrals (4.2), (4.6), being quadratic with respect to the velocities ($\boldsymbol{M}$ or $\boldsymbol{\omega}$), depend on the positional variables in a rather complex way. Maybe for this reason the classics (in particular, Routh and F. Noether who obtained only particular results) did not find these integrals. As it has been already noted, the integrals (4.2), (4.6) were originally found through numerical experiments. Their analytic form was obtained by means of the following considerations.

As we have shown above, the problem concerning the rolling of a ball on a paraboloid of revolution such that $b_1 = b_2$ is integrable and has two additional linear integrals. This integrals (in the absence of rotation $\Omega = 0$) may be written as follows

$$\begin{gathered} I_1 = \gamma_3^{\sqrt{1-k}-1}(\sigma_1 - \sigma_2\gamma_3 + \sigma_2\gamma_3\sqrt{1-k}), \\ I_2 = \gamma_3^{-\sqrt{1-k}-1}(\sigma_1 - \sigma_2\gamma_3 - \sigma_2\gamma_3\sqrt{1-k}), \quad k = \frac{D}{\mu + D} \end{gathered} \tag{4.7}$$

where $\sigma_1 = \omega_3$, $\sigma_2 = (\boldsymbol{\omega}, \boldsymbol{\gamma})$. The product of these integrals is also an integral which is quadratic with respect to σ_i, and a rational function of σ_1, σ_2 and γ_3

$$J = \frac{(\sigma_2\gamma_3 - \sigma_1)^2}{\gamma_3^2} - \sigma_2^2(1 - k) = \frac{\sigma_1^2}{\gamma_3^2} - 2\sigma_2\frac{\sigma_1}{\gamma_3} + k\sigma_2^2. \tag{4.8}$$

Let us eliminate, using the expression for energy (3.7), the term $k\sigma_2^2$ from the integral (4.8), and consider the integral

$$F_2 = J + 2E = \frac{\omega_3^2}{\gamma_3^2} - 2\sigma_2\frac{\omega_3}{\gamma_3} + \boldsymbol{\omega}^2. \tag{4.9}$$

Substituting the expressions for σ_2 into (4.9) and isolating perfect squares, we obtain

$$F_2 = \frac{(\gamma_2\omega_3 - \gamma_3\omega_2)^2 + (\gamma_3\omega_1 - \gamma_1\omega_3)^2}{\gamma_3^2}. \tag{4.10}$$

The integral F_2, written in such a form, is easily generalized to the case of an arbitrary paraboloid ($b_1 \neq b_2$)

$$F_2 = \frac{\dfrac{1}{b_1}(\gamma_2\omega_3 - \gamma_3\omega_2)^2 + \dfrac{1}{b_2}(\gamma_3\omega_1 - \gamma_1\omega_3)^2}{\gamma_3^2} = \frac{(\boldsymbol{\gamma} \times \boldsymbol{\omega}, \mathbf{B}^i(\boldsymbol{\gamma} \times \boldsymbol{\omega}))}{\gamma_3^2}, \quad \mathbf{B}^i = \operatorname{diag}(b_1^{-1}, b_2^{-1}, 0) \tag{4.11}$$

and any surface of the second order (4.2). The integrals (4.2), (4.6) may be used for stability analysis of stationary motions of a ball near the points of intersection of the surface with the principal axes. The symmetrical case of this problem was considered by Routh [10].

Motion of a ball on an elliptic cone. Suppose that the ball's center of mass is moving on the surface of an elliptic cone, defined by the equation

$$(\boldsymbol{r}_c,\, \mathbf{B}^{-1}\boldsymbol{r}_c) = 0, \qquad \mathbf{B} = \operatorname{diag}(b_1, b_2, -1), \tag{4.12}$$

where $\boldsymbol{r}_c = \boldsymbol{r} + R\boldsymbol{\gamma}$ are the coordinates of the center of mass, and b_1, b_2 are positive quantities such that $\sqrt{b_1}$ and $\sqrt{b_2}$ determine the slope of the generatrices with respect to the coordinate axes. Given the coordinates of the center of mass, we can calculate the normal to the surface $\boldsymbol{\gamma}$ at this point as follows:

$$\boldsymbol{\gamma} = \frac{\mathbf{B}^{-1}\boldsymbol{r}_c}{\sqrt{(\mathbf{B}^{-1}\boldsymbol{r}_c,\, \mathbf{B}^{-1}\boldsymbol{r}_c)}}. \tag{4.13}$$

In our case $\boldsymbol{r}$ (or $\boldsymbol{r}_c$) is not uniquely defined by $\boldsymbol{\gamma}$ (because $\boldsymbol{\gamma}$ is constant on a generatrix). Therefore, as phase variables, we will use not $(\boldsymbol{M},\, \boldsymbol{\gamma})$ (as we did earlier), but $(\boldsymbol{M},\, \boldsymbol{r}_c)$. Upon substitution of (4.13) into the equations of motion (2.3), we get the equations of motion in terms of the variables $\boldsymbol{M}$, $\boldsymbol{r}_c$:

$$\begin{cases} \dot{\boldsymbol{r}}_c = \dfrac{R}{(\mu + D)\sqrt{(\mathbf{B}^{-1}\boldsymbol{r}_c,\, \mathbf{B}^{-1}\boldsymbol{r}_c)}}(\boldsymbol{M} \times \mathbf{B}^{-1}\boldsymbol{r}_c), \\ \dot{\boldsymbol{M}} = \dfrac{DR}{(\mu + D)^2(\mathbf{B}^{-1}\boldsymbol{r}_c,\, \mathbf{B}^{-1}\boldsymbol{r}_c)^{5/2}}(\mathbf{B}^{-1}(\boldsymbol{M} \times \mathbf{B}^{-1}\boldsymbol{r}_c),\, \mathbf{B}^{-1}\boldsymbol{r}_c \times (\mathbf{B}^{-1}\boldsymbol{r}_c \times \boldsymbol{M}))\mathbf{B}^{-1}\boldsymbol{r}_c. \end{cases} \tag{4.14}$$

Equations (4.14) possess an energy integral

$$H = \frac{1}{2(\mu + D)}\Big(\boldsymbol{M}^2 + \frac{D(\boldsymbol{M},\, \mathbf{B}^{-1}\boldsymbol{r}_c)^2}{\mu(\mathbf{B}^{-1}\boldsymbol{r}_c,\, \mathbf{B}^{-1}\boldsymbol{r}_c)}\Big)$$

and an invariant measure

$$\rho = \sqrt{(\mathbf{B}^{-1}\boldsymbol{r}_c,\, \mathbf{B}^{-1}\boldsymbol{r}_c)}.$$

Let us now make one more change of variables and time

$$\boldsymbol{y} = \mathbf{B}^{-1}\boldsymbol{r}_c, \qquad d\tau = \frac{R}{(\mu + D)\sqrt{(\mathbf{B}^{-1}\boldsymbol{r}_c,\, \mathbf{B}^{-1}\boldsymbol{r}_c)}}dt.$$

This results in

$$\begin{cases} \boldsymbol{y}' = \mathbf{B}^{-1}(\boldsymbol{M} \times \boldsymbol{y}), \\ \boldsymbol{M}' = \dfrac{D}{(\mu + D)(\boldsymbol{y},\, \boldsymbol{y})^2}(\mathbf{B}^{-1}(\boldsymbol{M} \times \boldsymbol{y}),\, \boldsymbol{y} \times (\boldsymbol{y} \times \boldsymbol{M}))\boldsymbol{y} \end{cases} \tag{4.15}$$

which possesses two "natural" integrals: the energy integral

$$H = \frac{1}{2(\mu + D)}\Big(\boldsymbol{M}^2 + \frac{D}{\mu}\frac{(\boldsymbol{M},\, \boldsymbol{y})^2}{(\boldsymbol{y},\, \boldsymbol{y})}\Big),$$

and the geometrical integral

$$(\boldsymbol{y},\, B\boldsymbol{y}) = 0.$$

The latter defines the surface in terms of the new variables along which the ball's center of mass is moving. Moreover, equations (4.15) possess invariant measure with constant density. The generalization of the nontrivial integral (4.2) to the case of equations (4.15) looks like

$$F_2 = ((\boldsymbol{M} \times \boldsymbol{y}),\, \mathbf{B}^{-1}(\boldsymbol{M} \times \boldsymbol{y})), \tag{4.16}$$

or, in terms of the original physical variables, we set

$$F_2 = ((\boldsymbol{M} \times \mathbf{B}^{-1}\boldsymbol{r}_c),\, \mathbf{B}^{-1}(\boldsymbol{M} \times \mathbf{B}^{-1}\boldsymbol{r}_c)).$$

The question of existence of one more additional integral for equations (4.15) remains open. Apparently, in the general case, when $b_1 \neq b_2$, it does not exist.

5. Motion of a ball on a cylindrical surface

Let us consider the rolling of a ball whose center of mass is moving on a cylindrical surface. It can be shown that, in the absence of external fields, this system may be integrated by quadratures; but when an external force is directed along the cylinder generatrix, the equations are reduced to a Hamiltonian system with one and a half degree of freedom .

Let us choose a fixed frame of reference with one axis (Oz) directed along the cylinder generatrix (see Fig. 6). In this case, a normal vector is expressed as

$$\boldsymbol{\gamma} = (\gamma_1, \gamma_2, 0), \quad \gamma_1^2 + \gamma_2^2 = 1. \tag{5.1}$$

Denote the projections of the normal to the center of mass and the position vector of the center of mass of the ball onto the normal cross-section by $\widetilde{\boldsymbol{r}} = (r_1 + R\gamma_1, r_2 + R\gamma_2)$, $\widetilde{\boldsymbol{\gamma}} = (\gamma_1, \gamma_2)$. For these projections we have evident geometrical relations

$$(\dot{\widetilde{\boldsymbol{r}}}, \widetilde{\boldsymbol{\gamma}}) = (\dot{\widetilde{\boldsymbol{\gamma}}}, \widetilde{\boldsymbol{\gamma}}) = 0.$$

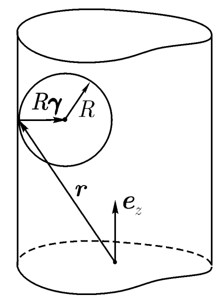

Fig. 6

Hence, we conclude that $\dot{\widetilde{\boldsymbol{\gamma}}}$ is parallel to $\dot{\widetilde{\boldsymbol{r}}}$,

$$\dot{\widetilde{\boldsymbol{\gamma}}} = \lambda(\boldsymbol{\gamma})\dot{\widetilde{\boldsymbol{r}}}.$$

The factor $\lambda(\boldsymbol{\gamma})$ is completely determined by the geometry of the cylinder cross-section and does not depend on angular velocity.

Using equations (2.3), we get the equations of motion for the ball on the cylindrical surface with the assumption that the cylinder is subject to a force (with potential $U(z)$) directed along a cylinder's generatrix ($z = \frac{r_3}{R}$):

$$\begin{gathered}
\dot{\boldsymbol{M}} = \frac{M_3}{\mu + D}\lambda(\boldsymbol{\gamma})\frac{D}{\mu + D}(\boldsymbol{M} \times \boldsymbol{\gamma}, \boldsymbol{e}_z)\boldsymbol{\gamma} + \frac{\partial U}{\partial z}\boldsymbol{e}_z \times \boldsymbol{\gamma}, \\
\dot{\boldsymbol{\gamma}} = \frac{M_3}{\mu + D}\lambda(\boldsymbol{\gamma})\boldsymbol{e}_z \times \boldsymbol{\gamma}, \quad \dot{z} = \frac{1}{\mu + D}(\boldsymbol{M} \times \boldsymbol{\gamma}, \boldsymbol{e}_z).
\end{gathered} \tag{5.2}$$

Here not only the energy is conserved but also the projection of the (angular velocity) moment on the cylinder axis:

$$\begin{gathered}
H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) + U(z), \\
F_2 = M_3 = (\mu + D)\omega_3 = \mathrm{const}.
\end{gathered} \tag{5.3}$$

Moreover, the system (5.3) possesses an invariant measure with density

$$\rho(\boldsymbol{\gamma}) = \lambda^{-1}(\boldsymbol{\gamma}). \tag{5.4}$$

It follows from (5.2) that the equations for the vector $\boldsymbol{\gamma}$ get uncoupled. Let us use parametrization

$$\gamma_1 = \cos\varphi, \quad \gamma_2 = \sin\varphi.$$

For the angle $\varphi(t)$ we obtain the equation

$$\dot{\varphi} = \frac{M_3}{\mu + D}\lambda(\cos\varphi, \sin\varphi) = Q^{-1}(\varphi), \tag{5.5}$$

where $Q(\varphi)$ is, in the general case, a 2π-periodic function of φ, determined by the shape of a cylinder cross-section.

In the remaining equations of the system (5.2) we put

$$K_1 = M_1\gamma_1 + M_2\gamma_1, \quad K_2 = M_1\gamma_2 - M_2\gamma_1,$$

and replace the time (as an independent variable) by the angle φ, and thereby obtain the nonautonomous system with 2π-periodic coefficients

$$\begin{gathered}\frac{dK_1}{d\varphi} = -\frac{\mu}{\mu + D}K_2, \quad \frac{dK_2}{d\varphi} = K_1 - Q(\varphi)U'(z),\\ \frac{dz}{d\varphi} = \frac{Q(\varphi)}{\mu + D}K_2.\end{gathered} \tag{5.6}$$

This system has an integral of energy

$$\widetilde{H} = \frac{1}{2}\Big(\frac{K_1^2}{\mu} + \frac{K_2^2}{\mu + D}\Big) + U(z). \tag{5.7}$$

In the case of the gravity field, $U(z) = mgz$ and equation (5.6) are integrated by quadratures:

$$\begin{gathered}K_1(\varphi) = -\nu mg\int\limits_{\varphi_0}^{\varphi}\sin\nu(\tau - \varphi)Q(\tau)d\tau + \nu A\cos\nu\varphi + \nu B\sin\nu\varphi,\\ K_2(\varphi) = -mg\int\limits_{\varphi_0}^{\varphi}\cos\nu(\tau - \varphi)Q(\tau)d\tau + A\sin\nu\varphi - B\cos\nu\varphi,\end{gathered} \tag{5.8}$$

where A, B are constants and $\nu^2 = \dfrac{\mu}{\mu + D}$.

Let us show that the integrals in (5.8) are bounded functions. We expand the function $Q(\tau)$ in a Fourier series

$$Q(\tau) = \sum_{n\in\mathbb{Z}} Q_n e^{in\tau}. \tag{5.9}$$

The integrals in the expressions for $K_1(\varphi)$ and $K_2(\varphi)$ in (5.8) may be considered as the real and imaginary parts of the integral

$$\int e^{i\nu\tau}Q(\tau)\,d\tau = \int\sum_n Q_n e^{i(n+\nu)\tau}\,d\tau. \tag{5.10}$$

Using the well known theorems of the Fourier analysis and the fact that $n+\nu \neq 0$ (as far as $0 < \nu < 1$) we put the integral under the sum sign and integrate the series term-by-term:

$$\int\sum_n Q_n e^{i(n+\nu)\tau}\,d\tau = \sum_n \frac{Q_n}{i(n+\nu)}e^{i(n+\nu)\tau}. \tag{5.11}$$

It is evident that the series obtained converges to a certain quasiperiodic function, hence, $K_1(\varphi)$ and $K_2(\varphi)$ are bounded. This fact and also the conservation of energy in the reduced system (5.7) ensure that $z(\varphi)$ is bounded.

Thus, when the ball is rolling on an absolutely rough cylindrical surface of an arbitrary cross-section in the gravity field, the vertical secular drift is not observed.

The time dependence of the angle (and, hence, of all the other functions) is described by (5.5).

An elliptic (hyperbolic) cylinder. Let us consider in greater detail a particular case, i. e., the ball's center of mass is moving on an elliptic cylinder with cross-section is defined by the equation

$$\frac{x^2}{b_1} + \frac{y^2}{b_2} = 1. \tag{5.12}$$

We have

$$r_1 + R\gamma_1 = \frac{b_1\gamma_1}{\sqrt{b_1\gamma_1^2 + b_2\gamma_2^2}}, \quad r_2 + R\gamma_2 = \frac{b_2\gamma_2}{\sqrt{b_1\gamma_1^2 + b_2\gamma_2^2}}, \quad r_3 = Rz,$$

therefore,

$$\lambda(\boldsymbol{\gamma}) = \frac{R(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})^{3/2}}{b_1 b_2}, \quad \mathbf{B} = \operatorname{diag}(b_1, b_2, 0),$$
$$Q^{-1}(\varphi) = \frac{M_3 R}{(\mu + D) b_1 b_2}(b_1 \cos^2\varphi + b_2 \sin^2\varphi)^{3/2}.$$

We should mention an important distinction existing between an elliptic and a circular cylinder (see above): in the case of an elliptic cylinder the dependency of dynamical variables K_1, K_2, z is defined by two frequencies $\omega_1 = 1$, $\omega_2 = \nu$, instead of a single frequency, as it happens in case of a circular cylinder. Thus, the integrals in (5.8) contain quasiperiodic functions; the integrals have very complicated nature, their analytical properties are thoroughly discussed in [7]. Graphs $z(\varphi)$ for various initial values of K_1 and K_2 are shown in Fig. 7. The main result is that whatever the ratio of the frequencies, the quantities K_1 and K_2, and, therefore, the displacement z execute bounded, quasiperiodic oscillations. It is the main result of this construction.

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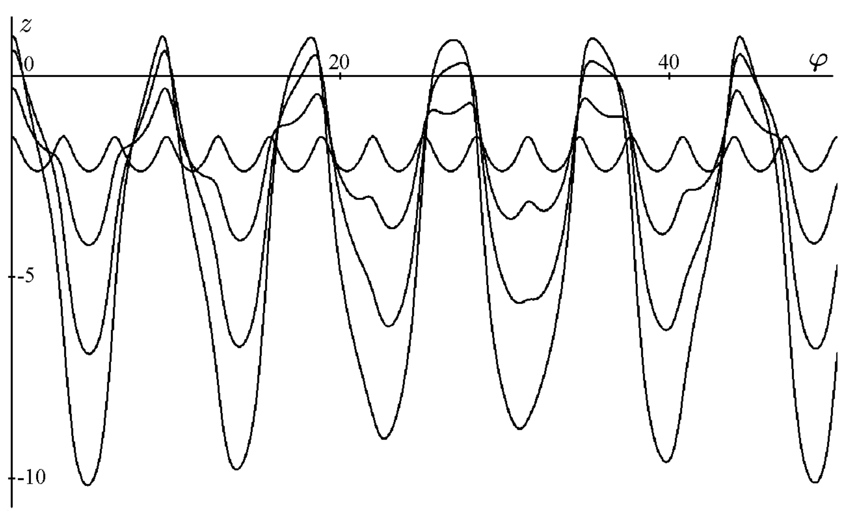

Fig. 7. The φ-dependence of the vertical coordinate of the point of contact z for various initial values K_1, K_2, z. For this figure the other parameters are: $E = 1$, $\mu = 1$, $D = 1 (\nu = 2^{-1/2})$, $b_1 = 1$, $b_2 = 2$, $R = 1$.

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Table 1. The rolling of a ball on a surface

		surface of the second order		surface of revolution			
surface type	cylindrical surface	ellipsoid, hyperboloid, paraboloid	cone of the second order	arbitrary surface	ellipsoid, hyperboloid	paraboloid, cone, cylinder	sphere
measure	measure exists	$\rho = (\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})^{-2}$	$\rho = \sqrt{(\mathbf{B}^{-1}\boldsymbol{r}_c, \mathbf{B}^{-1}\boldsymbol{r}_c)}$	$\rho = (f(\gamma_3))^3\Big(f(\gamma_3) - \dfrac{1-\gamma_3^2}{\gamma_3}f'(\gamma_3)\Big)$			
additional integrals	system is integrable by quadratures	$\dfrac{(\boldsymbol{\gamma}\times\boldsymbol{M}, \mathbf{B}^{-1}(\boldsymbol{\gamma}\times\boldsymbol{M}))}{(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})} = \mathrm{const}$ (one integral)	$((\boldsymbol{M}\times\mathbf{B}^{-1}\boldsymbol{r}_c), \mathbf{B}^{-1}(\boldsymbol{M}\times\mathbf{B}^{-1}\boldsymbol{r}_c)) =$ $= \mathrm{const}$ (one integral)	two linear integrals, defined by the system of linear equations	there exist two linear integrals that can be expressed in terms of elementary functions		
Hamiltonianity	nothing is known about the Hamiltonianity of these systems			upon change of time (prescribed by the reducing multiplier) the reduced system becomes Hamiltonian			
authors	A.V.Borisov, I.S.Mamaev, A.A.Kilin (2001)			E.Routh (1884)	A.V.Borisov, I.S.Mamaev, A.A.Kilin (2001)	E.Routh (1884)	
generalizations and remarks	integrable addition of the gravity field along a cylinder generatrix is possible			A.V.Borisov, I.S.Mamaev, and A.A.Kilin have shown the integrability in terms of elementary functions of the case of the ball rolling on the ellipsoid of revolution; have found an invariant measure for an arbitrary surface of revolution; for a paraboloid, a cone, and a cylinder they have shown the integrability of the case, when the ball is rolling on a surface rotating around its axis of symmetry			the problems of the rolling of a ball on an unconstrained and rotating sphere are also solved. System also allows integrable additions of potentials.

Remark. The cases when the tensor invariants exist are indicated by gray color in the table. The partial filling corresponds to the uncomplete set of integrals.